Tutorial 9

Gidon Rosalki

2025-05-28

1 Verifiers

Reminder: We define the language NP as follows:

$$NP = \bigcup_{k=0}^{\infty} NTIME\left(n^k\right)$$

Definition 1.1 (Polynomial verifier)**.** *A verifier for a language L is a deterministic TM V which given (w, c), with c polynomial in $|w|$, runs in polynomial time in $|w|$ and*

$$\forall w \in L,\ \exists c \in \Sigma^* : V(w, c) = q_{acc}$$
$$\forall w \notin L,\ \forall c \in \Sigma^*\ V(w, c) = q_{rej}$$

This c is called a witness.

Theorem 1 (Equivalent definition of NP)**.** *A language L can be recognized by an NTM in polynomial time* ***if and only if*** *there is a polynomial verifier for L.*

Proof . Proved in lecture 9 □

Example 1. *Let*

$$\textit{U-ST-HAMPATH} = \{\langle G, s, t\rangle : G \textit{ is an undirected graph, and there exists a Hamiltonian path from } s \textit{ to } t\}$$

Show that U-ST-HAMPATH is NP-Complete:

Solution. We first need to show that U-ST-HAMPATH is in NP: We will show that there exists a polynomial time verifier, V. Let V take as input $w = \langle G, s, t\rangle$, and a witness $c = \langle v_1, \ldots, v_n\rangle$, a sequence of $n = |V|$ vertices. V then performs the following:

1. Check if the first vertex is s, and the last is t
2. Check that all nodes are different (since if this is not the case, then there is a repetition, and this is not a Hamiltonian path)
3. Validate the existence of an edge between every two adjacent vertices

$\langle G, s, t\rangle \in U-ST-HAMPATH$ **if and only if** There is a Hamiltonian path in G, from s to t **if and only if** There exists $c \in \Sigma^*$ such that V accepts (w, c).
The witness c is polynomial in the size of the input $|\langle G, s, t\rangle|$, and each operation can be performed in polynomial time.

We now need to show that U-ST-HAMPATH is NP-Hard. In the previous tutorial, it was shown that $3-SAT \leq_p D-ST-HAMPATH$. As 3-SAT is NP-Hard, then so is D-ST-HAMPATH. We can thus show that U-ST-HAMPATH is NP-Hard, by performing the reduction $D-ST-HAMPATH \leq_p U-ST-HAMPATH$. Let there be the reduction $f(\langle G, s, t\rangle) = \langle G', s_{in}, t_{out}\rangle$, where G' is defined as follows:

- For every vertex $v \in V$, our function will write the new vertices v_{in}, v_{mid}, v_{out}, and connect the nodes with the following edges: $\{v_{in}, v_{mid}\}, \{v_{mid}, v_{out}\}$.
- For every edge $(u, v) = e \in E$, we will define the new edge u_{out}, v_{in}.

Correctness: Let us assume that $\langle G, s, t\rangle \in D-ST-HAMPATH$. Then, let there be the directed Hamiltonian path $s, u^1, u^2, \ldots, u^k, t$ in G. Therefore, there is the path in G':

$$s_{in}, s_{mid}, s_{out}, u^1_{in}, u^1_{mid}, u^1_{out}, \ldots u^k_{in}, u^k_{mid}, u^k_{out}, t_{in}, t_{mid}, t_{out}$$

Since the original path contained all the vertices, then so too does the new path in G', thus

$$\langle G', s, t\rangle \in U-ST-HAMPATH$$

Let there be a Hamiltonian path in G' from s_{in} to t_{out}. Thus, there is a Hamiltonian path in G', but we do not know what this path "looks like". We proceed with the following claim. A Hamiltonian path that starts at s_{in} and ends at t_{out} does not contain a directed traversal of the form (v_{in}, u_{out}). That is to say, we never travel "backwards" along an edge.
Let us assume that there is such a directed traversal, and let (v_{in}, u_{out}) be the first such one in this path. Since the path is Hamiltonian, we must visit v_{mid} at some point on the path. If we have already visited it, then we must have visited it from v_{out}, since we are now located at v_{in}, and it has no other edges, but in that case, in order to reach v_{out}, we must have arrived from some x_{in}, which is a contradiction to (v_{in}, u_{out}) being the first such pair. Therefore, we must visit v_{mid} after visiting u_{out}, and therefore we must reach it from v_{out}, at which point we get stuck, since we have already visited v_{in}. Therefore, we cannot reach t_{out}. We can therefore conclude that all the edges in the path are of the formats $(v_{in}, v_{mid}), (v_{mid}, v_{out}), (v_{out}, u_{in})$. Thus, the Hamiltonian path is of the form

$$s_{in}, s_{mid}, s_{out}, u^1_{in}, u^1_{mid}, u^1_{out}, \dots u^k_{in}, u^k_{mid}, u^k_{out}, t_{in}, t_{mid}, t_{out}$$

which can be easily mapped to a Hamiltonian path in G.

Runtime: Clearly polynomial, since $|G'| = 3|G|$, and computing it is trivial. □

2 3-COLOURING

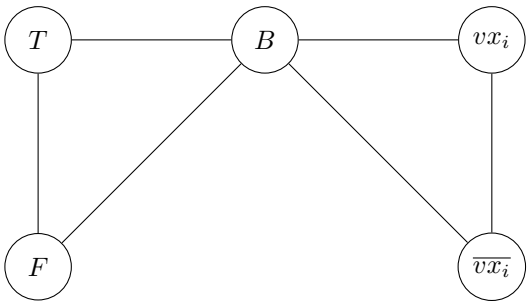
Given a graph $G = (V, E)$, the k-COLOURING problem is to validate whether or not it is possible to colour all vertices in k different colours such that no 2 neighbouring vertices have the same colour.

$$\text{3-COLOURING} = \{\langle G\rangle : \text{ There's a valid 3-colouring of } G\}$$

One way to check if a graph $\langle G\rangle$ is not in 3-COLOURING is by validating the existence of a clique of size 4 in G. This can be done in polynomial tie, since there are $\binom{|V|}{4} = O\left(|V|^4\right)$ sets of size 4, and it takes polynomial time to verify if each is a clique.

Theorem 2 (3-COLOURING is NP-Complete)**.** *We will first show that $3C \in NP$ by using a verifier V. Given a graph $\langle G\rangle$, and a colouring c, V checks if c is a valid colouring of G, by going over all the edges in E, and verifying that the vertices at each edge have different colours. This can be done in polynomial time.*
We will now prove that $3 - SAT \leq_p 3C$, thus proving hardness in NP. Let $\langle\varphi\rangle$ be a 3-CNF formula, with $x_1, \dots, x_n$ variables, and $c_1, \dots, c_m$ clauses. The reduction creates the following graph:

1. *Create 3 vertices, T, F, B, and connect them all together.*
2. *For each variable $x_i \in \varphi$, create 2 vertices, $vx_i, \overline{vx_i}$, add an edge between them, and an edge from each of them to B.*



3. *For each clause $c_j = (l_1 \vee l_2 \vee l_3)$, add triple or gadget graph, connect to relevant literals vertices, and connect the output to B and to F (forcing it to be T)*

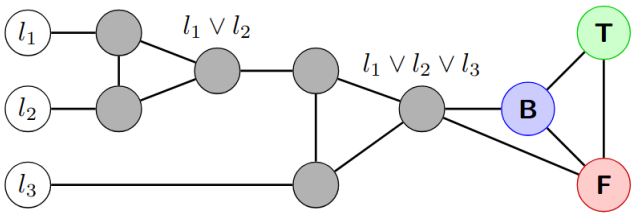


Figure 1:

Notice, if l_1, l_2, l_3 are all false, then the triple or gadget output must be coloured false, but it is also connected to F , so it is an invalid colouring. On the other hand, if at least one of them is coloured true then there exists a valid colouring.

Correctness: $\varphi \in 3SAT \implies \varphi$ *is satisfiable. Let* $A(x_i) \to \{T, F\}$ *be a satisfying assignment. If* x_i *is assigned* T*, we colour* vx_i *green* $= T$*, and* $\overline{vx_i}$ *red* $= F$*. Otherwise, we assign the opposite. This is valid as they are connected to* B*, and to each other, 3 different colours so far. Since* φ *is satisfiable, each clause* c_j *is satisfiable, so at least one of the literals is true, and from the gadget observation, it is 3-colourable.*
Let $\langle G \rangle \in 3C$*. We construct a satisfying assignment through the colour of the vertices* vx_i*. If it is coloured true we assign true, otherwise false. If by contradiction this is not a satisfying assignment, then there is a clause* c_j *which is not satisfied. As such,* l_1^j, l_2^j, l_3^j *are all false, and are thus all coloured false in the graph. In this case this means the clause triple or gadget is not 3 colourable, thus the entire graph is not 3 colourable in contradiction. So the assignment must be a satisfying assignment.*
Runtime: For each variable we construct 2 nodes, and connect 3 edges, this is polynomial in n*. For each clause we create an or gadget which is a fixed size of nodes and edges. There are* m *gadgets, so this too is polynomial in the input size.*